

On rotation and rotating frames: I-Franklin transformation and its modification

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Abstract

Unlike the Lorentz transformation which replaces the Galilean transformation among inertial frames at high uniform velocities, there seems to be no such a consensus in the case of rotating frames and the common practice is the use of the classical Galilean rotational transformation. There has been some attempts to generalize this transformation to high rotational velocities (i.e high angular velocities or large radial distances). Here we introduce a modified version of one of these transformations proposed by Philip Franklin in 1922, which is shown to resolve some of its shortcomings specially with respect to the corresponding spacetime metric in the rotating frame. The modified transformation is reinterpreted in terms of non-inertial observers sitting at non-zero radii and the corresponding metric in rotating frame is shown to be consistent with the one obtained through Galilean rotational transformation for points close to the axis. Spatial distances and time intervals based on the same spacetime metric are also discussed.

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I. INTRODUCTION

“*There is no relativity of rotation*”. This relatively famous quote by Feynman [1] may look as the final word on the discussion of rotation in the context of special relativity. Based on the fact that the presence of acceleration in a uniformly rotating frame, by the equivalence principle, takes us into the realm of general relativity may convince one not to bother with the formulation of rotation in the context of special relativity and look for the resolution of each rotation-based problem in general relativity and in the suitably chosen/constructed solutions of Einstein field equations (which are of course not usually available). On the other hand rotation and rotating frames have always been a source of confusion [17], the famous example of which is the *Ehrenfest’s Paradox* [3]. It seems that this same paradox is a good opening point to start our discussion on rotation and rotational transformations. To explain the paradox we consider two frames one at rest (the laboratory observer) and the other one rotating counter-clockwise around it with constant angular velocity Ω (the rotating observer). Using cylindrical coordinates we denote the non-rotating frame with coordinates (t, r, ϕ, z) and the one rotating around the z –(z' –) axis with (t', r', ϕ', z') . These are related through the Galilean rotational transformation (GRT)

$$t' = t \quad , \quad r' = r \quad , \quad \phi' = \phi - \Omega t \quad , \quad z' = z \quad (1)$$

or in its differential form

$$dt' = dt \quad , \quad dr' = dr \quad , \quad d\phi' = d\phi - \Omega dt \quad , \quad dz' = dz \quad (2)$$

Through the above equation we would like to emphasize on the meaning of the Galilean rotational transformation which, as in the case of linear Galilean and Lorentz transformations, introduces a prescription of how the spacetime coordinates of *an event* in the two frames are related. This interpretation leads to the following relation between the angular velocities of a test particle observed in the two frames (Figure 1)

$$\omega' = \omega - \Omega \quad (3)$$

In other words, in this interpretation, (1) is a *kinematical transformation* between two frames and not one particular to the problem of a rotating disk, though it could also be employed for a rotating disk and its points observed by both the laboratory observer and the rotating

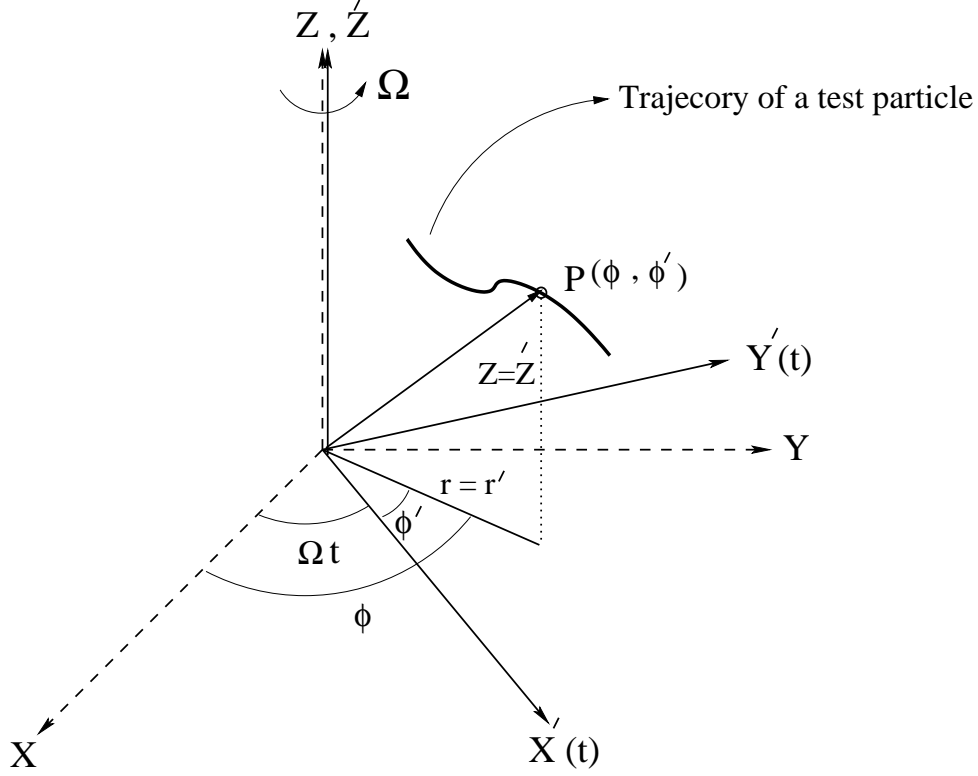


FIG. 1: Two frames, one rotating (solid) around the other (dashed) with uniform angular velocity Ω . Trajectory of a test particle and a point P on it as an event observed in the two frames, assigned with angular velocities ω and ω' .

observer attached to the axis of the disk. In this case it is expected that for any point on the disk $\omega' = 0$ and $\omega = \Omega$ (Figure 2). This differentiation between a kinematical rotational transformation and the one tailored for a rotating disk is an important point which should be taken into account when discussing interpretations of any relativistic rotational transformation including the ones we are going to consider in the following sections (see section IIA).

II. EHRENFEST'S PARADOX

Ehrenfest's paradox is a contradiction that *an inertial observer* faces in applying special relativistic length contraction to a rotating disk. From an inertial observer's point of view the rim of a rotating disk undergoes a length contraction due to its transverse motion with velocity $v = R\Omega$ and so circumference of a rotating disk (P') is shorter than the one non-

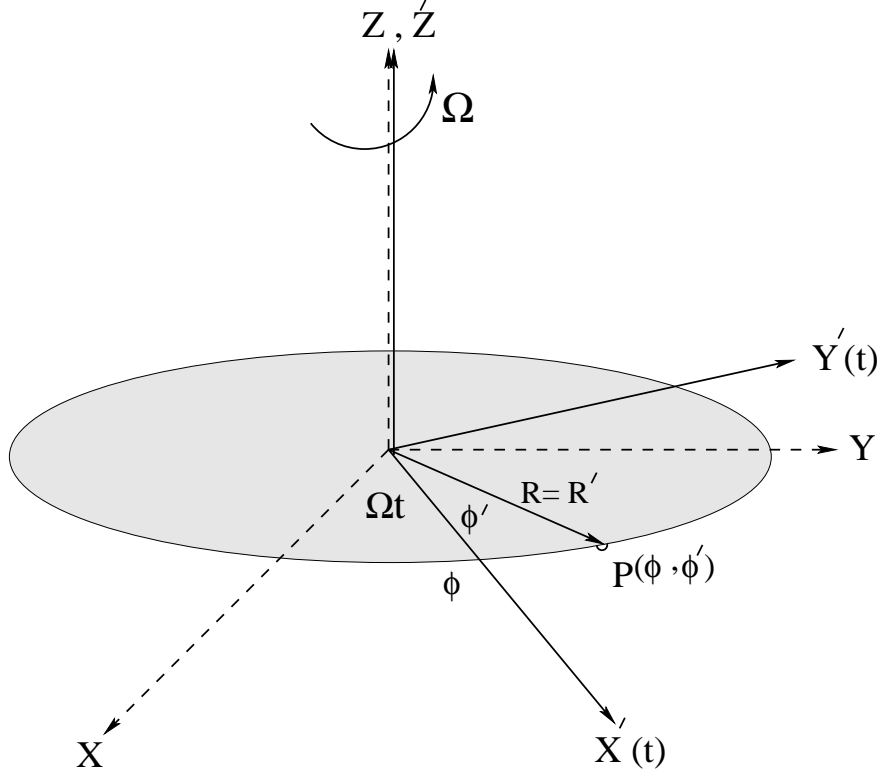


FIG. 2: A disk and its frame (solid) rotating around the laboratory frame (dashed) with uniform angular velocity Ω . Coordinates of a point P in the rim are given in the two frames with the assigned angular velocities $\omega' = 0$ and $\omega = \Omega$.

rotating (P), i.e $P' < P$. On the other hand since the radius of the disk is perpendicular to the rotational motion of the rim it will not experience contraction and so $R' = R$. Therefore the same observer, living in a flat spacetime and thereby using the *Euclidean* prescription for the circumference of a circle, finds out the contradicting result $P = 2\pi R = 2\pi R' = P'$. Perhaps it should be left for experiment to decide which relation holds between P and P' but nevertheless people have tried hard to find either a theoretical resolution to this paradox or otherwise to invalidate it. An apparently favorite resolution in the literature is based on considering the situation from the point of view of a rotating observer (frame) and on the simple fact that the spatial geometry of a rotating frame is *non-Euclidean*. But, as we will show below, that does not seem to be leading to any kind of resolution of the paradox but to a somewhat similar paradox from the rotating observer's point of view. In the case of a rotating disk one should distinguish between the observer at the center (*spinning observer* at $r = 0$) and those at different non-zero radii which are non-inertial due to the centrifugal

acceleration [18](*orbiting observers*). Later on rotating observers at non-zero radii will be of central importance in our discussion of relativistic rotational transformations but for the purpose of Ehrenfest paradox we only deal with the rotating observer at the center. From the point of view of a rotating observer the above mentioned non-Euclidean character of disk geometry could be obtained from considering the metric of flat spacetime in the rotating frame, as it is the spatial geometry (metric), defined through spacetime metric, which accounts for spatial distances including that of the disk circumference. Using the differential Galilean rotational transformation (2), the flat spacetime metric in the non-rotating frame

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2 \quad (4)$$

transforms into

$$ds^2 = (c^2 - \Omega^2 r^2) dt^2 - 2\Omega r^2 dt d\phi' - dr^2 - r^2 d\phi'^2 - dz^2 \quad (5)$$

in the rotating frame. It is seen that this metric is applicable for radii less than c/Ω , corresponding to the so called *light cylinder*, beyond which g_{00} becomes negative (with the corresponding disk points having velocities greater than c) and hence from physical point of view not of interest.

The famous result, based on special relativistic arguments, that a rotating clock at non-zero radius $r = R$ runs slower than that sitting at the center of the disk is clearly encoded in the above metric, from which we have $d\tau = \sqrt{1 - \frac{\Omega^2 R^2}{c^2}} dt$ where dt is the world time recorded by the inertial clocks synchronized with the rotating one at the center. The above spacetime plays the same role for a rotating (*spinning*) observer that *Rindler space*

$$ds^2 = \eta_{ab} dx^a dx^b = (1 + a\bar{x}^1)^2 (d\bar{x}^0)^2 - (d\bar{x}^1)^2 - (d\bar{x}^2)^2 - (d\bar{x}^3)^2 \quad (6)$$

with

$$\begin{aligned} x^0 &= (a^{-1} + \bar{x}^1) \sinh(a\bar{x}^0) & ; & & x^2 &= \bar{x}^2 \\ x^1 &= (a^{-1} + \bar{x}^1) \cosh(a\bar{x}^0) & ; & & x^3 &= \bar{x}^3 \end{aligned} \quad (7)$$

plays for a uniformly accelerating observer with acceleration $a = (a_b a^b)^{\frac{1}{2}}$ [19]. It should be noted that the spacetime of a rotating observer (5), like Rindler space, is the flat spacetime in disguise (or more precisely not maximally extended) whose spatial line element,

defined through 1 + 3 formulation of spacetime decomposition (see appendix A for a brief introduction), is given by [14],

$$dl^2 = dr^2 + dz^2 + \frac{r^2 d\phi'^2}{1 - \frac{\Omega^2 r^2}{c^2}}. \quad (8)$$

Now for a circle of radius $r = r' = R$ in the $z = \text{constant}$ plane the circumference is given by

$$P' = \int_0^{2\pi} dl = \frac{2\pi R}{\sqrt{1 - \Omega^2 R^2/c^2}} = \frac{P}{\sqrt{1 - \Omega^2 R^2/c^2}} \quad (9)$$

so that $P' > P$ with P the circumference of a non-rotating disk. Therefore from the rotating observer's point of view P and P' are also not equal but the relation between the two quantities is just the opposite of that found by the laboratory observer based on Lorentz contraction. Interpretation of the above results goes as follows: Although the transformed spacetime is the flat spacetime in disguise, its spatial geometry now has non-zero Gaussian curvature leading to the fact that the ratio of the circumference of a circle to its radius is larger than 2π . We are not going to follow this disagreement on the relation between P and P' from the two observers' points of view nor discuss further the content of Ehrenfest's paradox but there remains a legitimate question that one might ask and that is:

Are we allowed to use the Galilean rotational transformation (1) in all the above considerations? specially noting that the metric in the rotating frame can be employed out to a specific radius given by c/Ω where it decreases as we increase the angular velocity. Our experience with Lorentz transformations intuitively leads to the expectation that Galilean rotational transformation to be an approximation valid for points near the axis of rotation having small linear velocities. Hence for points [20] far away from the axis one needs to replace the Galilean rotational transformation with a relativistic (Lorentz-type) rotational transformation to account for linear velocities comparable to c . Obviously if one could devise such a relativistic rotational transformation, it might be expected that either the transformation (based on its kinematical interpretation) or the spatial line element of the transformed flat space metric lead to a contracted/dilated circumference for a rotating disk or any other circle of a given radius. A comparison between the usual Lorentz transformation (LT), and the Galilean rotational transformation (GRT) is useful at this point. In the case of LT the length contraction is built in into the transformation itself and since the flat spacetime line element is form-invariant under the transformation, the length contraction is not expected to

be tractable in the form of the corresponding spatial metric. On the other hand in the case of GRT as we noticed, the transformation (1) is devoid of any length contraction or dilation while the transformed spatial metric (8) leads to the length dilation. One such relativistic rotational transformation was proposed by Franklin in 1922 [5] and some 30 years later by Trocheris [6] and Takeno [7] [21]. In the present article we will discuss this transformation and its properties including its advantages over the classical transformation and also its drawbacks specially with respect to the equivalent metric and show how a simple modified version of the transformation could lead to the resolution of some of these problems. For the sake of completeness we will give a brief derivation of Franklin transformation in the next section.

III. RELATIVISTIC ROTATIONAL TRANSFORMATION (FRANKLIN TRANSFORMATION)

Taking two coordinate frames S and S' , with S' uniformly rotating about S , Franklin requires the following plausible properties to be valid between the two frames [5]:

1-The velocity of a fixed point in S' with respect to the point in S with which it momentarily coincides is independent of the time, and is the same for all points at a given distance from the axis of rotation.

2-For the two concentric circles $r' = r = \text{Constant}$, the equations of transformation are similar to those for a Lorentz boost (say along the x-direction) with $r\phi$ the arc length replacing the linear distance (say x).

These two properties lead to the following transformation law

$$\begin{aligned} t' &= \gamma(r) (t - v(r)r\phi/c^2) & ; & \quad r' = r \\ r'\phi' &= \gamma(r) (r\phi - v(r)t) & ; & \quad z' = z \end{aligned} \tag{10}$$

in which $\gamma = \frac{1}{\sqrt{1-v(r)^2/c^2}}$ is the Lorentz-type factor with velocity $v(r)$ to be determined through the last property which is;

3-The velocity of a point at the distance $r' + \Delta r'$ from the axis with respect to a point at the distance r' from the axis (both in the system S') is given by $\Omega\Delta r'$. In other words two different observers at two different radii with two different rotational velocities are taken as the analogues of two inertial frames moving uniformly with respect to one another in usual

LT.

In effect, the last property is a prescription for velocity composition law, out of which the nontrivial form of the *rotational* velocity should be obtained. For two observers B and C sitting at radii $r_B = r$ and $r_C = r + \Delta r$ with velocities $v(r)$ and $v(r + \Delta r)$ (with respect to the inertial observer A at the center) respectively the composition law reads

$$v_{BC} = \frac{v_{AC} - v_{AB}}{1 - \frac{v_{AC}v_{AB}}{c^2}} \Rightarrow \Omega\Delta r = \frac{v(r + \Delta r) - v(r)}{1 - \frac{v(r + \Delta r)v(r)}{c^2}} \quad (11)$$

In the limit $\Delta r \rightarrow 0$ this leads to the velocity relation

$$v(r) = c \tanh(\Omega r/c) \quad (12)$$

Substituting (12) in (10), explicit form of the Franklin transformation is given by

$$\begin{aligned} t' &= \cosh(\Omega r/c)t - \frac{r}{c} \sinh(\Omega r/c)\phi \quad ; \quad r' = r \\ \phi' &= \cosh(\Omega r/c)\phi - \frac{c}{r} \sinh(\Omega r/c)t \quad ; \quad z' = z \end{aligned} \quad (13)$$

For points close to the axis i.e when $\frac{\Omega r}{c} \ll 1$ [22] this transformation reduces to the classical Galilean transformation by neglecting terms of order $\frac{\Omega^2 r^2}{c^2}$ and higher. These transformations form a group and the inverse transformation is given by changing $\Omega \rightarrow -\Omega$. One of the advantages of this transformation over the old Galilean one is in the definition of the velocity given in (12) which approaches c at $r \rightarrow \infty$ (i.e the light cylinder is not at a finite distance but is sent to infinity) and reduces to the Newtonian value $v = \Omega r$ for points near the axis. Formal Comparison with a pure Lorentz transformation as a hyperbolic rotation, reveals that, it is the linear velocity $v = \Omega r$ in (12) which now plays the role of some kind of *rapidity*.

Another obvious difference between Franklin transformation and Lorentz transformation is the fact that velocity entering the definition of FT unlike LT is not a constant but an r -dependent quantity. This will lead to undesirable results in the case of Franklin transformation when we consider both the transformed spacetime metric and the corresponding spatial distances and time intervals. It will be shown that neither will reduce to its expected Galilean expression for points near the axis (i.e when $\frac{\Omega r}{c} \ll 1$). But before discussing these issues, it seems appropriate to discuss interpretation of FT as compared to GRT and LT.

A. Interpretation of FT

An important issue about the Franklin transformation which seems to be taken for granted in most of the previous studies, is its interpretation as the transformation of the spacetime coordinates of an *event* between two frames; a non-rotating (inertial) frame and another one rotating about their common axis. This is the same usual interpretation employed for the GRT as illustrated in figure 1. But characteristics of FT would prevent one to easily interpret this transformation as a kinematical one. The main characteristic is the radial dependence of velocity entering the transformation. This velocity distribution is attributed to disk points and so, taking into account the fact that in FT the non-rotating and rotating frames share the same axis, the transformation of arc lengths in FT (which is given in terms of this velocity) is only valid for disk points. By the above reasoning, it seems more reasonable to look at FT as a transformation specially tailored for the problem of a rotating disk in which events are nothing but different points of a rotating disk at different times. In other words one should be cautious in interpreting FT as a kinematical transformation relating coordinates of an event in a rotating frame to that of an inertial non-rotating one. Further restriction of the transformation to the $z = \text{constant}$ plane and the fact that for $r = r' = 0$, i.e for an event (based on a kinematical interpretation) at the origin where the cylindrical coordinate system is degenerate and $v(0) = 0$, FT reduces *exactly* to GRT justifies this claim. If FT is going to be elevated to a kinematical transformation one needs to modify and reinterpret it. This will be done later when we introduce the modified Franklin transformation (MFT).

IV. SPACETIME METRIC AND SPATIAL GEOMETRY IN THE ROTATING FRAME THROUGH FRANKLIN TRANSFORMATION

Using the inverse of the Franklin transformation in its differential form

$$\begin{aligned}
cdt &= \cosh(\Omega r/c)cdt' + r \sinh(\Omega r/c)d\phi' + A_1 dr \quad ; \quad dr = dr' \\
rd\phi &= \cosh(\Omega r/c)rd\phi' + \sinh(\Omega r/c)cdt' + A_2 dr \quad ; \quad dz = dz' \\
A_1 &= \sinh(\Omega r/c)(\phi' + \Omega t') + \cosh(\Omega r/c)\left(\frac{\Omega r}{c}\phi'\right) \\
A_2 &= \sinh(\Omega r/c)\left(\frac{\Omega r}{c}\phi' - ct'/r\right) + \cosh(\Omega r/c)(\Omega t')
\end{aligned} \tag{14}$$

and substituting in (5) the spacetime metric in the rotating frame is given by

$$ds^2 = c^2 dt'^2 - (1 - A_1^2 + A_2^2) dr^2 - r^2 d\phi'^2 - dz^2 + 2c(A_1 \cosh(\Omega r/c) - A_2 \sinh(\Omega r/c)) dr dt' + 2(A_1 \sinh(\Omega r/c) - A_2 \cosh(\Omega r/c)) r dr d\phi' \quad (15)$$

Unlike the cross term in (5) which is the typical $dt'd\phi'$ term representing the rotation the cross terms in the above metric include $dr dt'$ and $dr d\phi'$ terms and that is why it is not expected that this metric reduce to (5) for $\Omega r/c \ll 1$ [11]. Further also it should be noted that due to the explicit appearance of ϕ' and t' in (15) both the *temporal* and *angular* isometries present in (5) are now lost.

A. Spatial distances and time intervals

From the above result on the spacetime metric it is obviously not expected that the spatial geometry corresponding to (15) be reducible to the one given by (8) in the limit $\Omega r/c \ll 1$. Indeed using the 1 + 3 decomposition (equation (A2)) the spatial metric corresponding to (14) is given by

$$dl^2 = \{1 - A_1^2 + A_2^2 + 4c^2[A_1 \cosh(\Omega r/c) - A_2 \sinh(\Omega r/c)]^2\} dr^2 + -2[A_1 \sinh(\Omega r/c) - A_2 \cosh(\Omega r/c)] r dr d\phi' + dz^2 + r^2 d\phi'^2 \quad (16)$$

through which the circumference of a disk with radius $r = R$ in the $z = \text{constant}$ plane is given by the Euclidean value $2\pi R$ compared to the non-Euclidean value (9) obtained through the Galilean transformed spatial metric (8). It should be noted that despite the above fact the Gaussian curvature of the spatial metric is not zero indicating the non-Euclidean nature of the spatial metric [5]. It should also be noted from (15) that proper time interval in the rotating frame is given by

$$d\tau' = c dt' = c \cosh(\Omega r/c) dt = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} dt \quad (17)$$

which in the limit $\frac{\Omega R}{c} \ll 1$ is different from the relation between the two time intervals obtained from the Galilean transformed metric for rotating clocks at nonzero radii i.e $d\tau' = \sqrt{1 - \frac{\Omega^2 r^2}{c^2}} dt$. On the other hand as we discussed earlier one could relate spatial distances and time intervals not through the metric obtained from Franklin transformation but in the coordinate transformations themselves according to their kinematical interpretation.

Obviously using the formal analogy between FT and LT one can obtain relation between spatial distances (arc lengths) and time intervals in the two coordinate systems as follows

$$\Delta t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta t \quad (18)$$

$$\Delta l' = r \Delta \phi' = \sqrt{1 - \frac{v^2}{c^2}} R \Delta \phi = \sqrt{1 - \frac{v^2}{c^2}} \Delta l \quad (19)$$

corresponding to the time dilation and length contraction respectively. With $v = c \tanh(\frac{\Omega R}{c})$ at radius $r = R$, the above results are consistent with what one expects from applying special relativistic length contraction (based on LT) to a rotating disk for $\frac{\Omega R}{c} \ll 1$. It seems that once again we are faced with the Ehrenfest's paradox, in the sense that using the geometry (either spatially Euclidean flat space of an inertial observer or spatially non-Euclidean flat space of a rotating observer given by equation (16)) tells us that the circumference of a rotating disk is the same as the one non-rotating whereas employing Franklin transformation, the circumference of a rotating disk is found to be shorter than the one non-rotating.

B. Angular velocity of a test particle/disk point in the two frames using FT

Using the differential Franklin transformation (14) to calculate the rotational frequency in the inertial observer's frame we find

$$\omega = \frac{d\phi}{dt} = \frac{\cosh(\Omega r/c) d\phi' + \frac{cdt'}{r} \sinh(\Omega r/c) + \frac{A_2}{r} dr'}{\cosh(\Omega r/c) dt' + \frac{r}{c} \sinh(\Omega r/c) d\phi' + \frac{A_1}{c} dr'} \quad (20)$$

from which for the frequency in the rotating frame we have

$$\omega' = \frac{d\phi'}{dt'} = \frac{\omega \cosh(\Omega r/c) - \frac{c}{r} \sinh(\Omega r/c) + \frac{dr}{dt'} (\frac{A_1}{c} \omega - \frac{A_2}{r})}{\cosh(\Omega r/c) - \omega \frac{r}{c} \sinh(\Omega r/c)} \quad (21)$$

In the classical limit where $(\Omega r/c) \ll 1$, the above expression reduces to the classical relation (3)

$$\omega' \approx \omega - \Omega. \quad (22)$$

V. MODIFIED FRANKLIN TRANSFORMATION, ITS INTERPRETATION AND THE EQUIVALENT METRIC IN ROTATING FRAME

As it is obvious from its derivation, Franklin transformation was obtained in close analogy with the usual Lorentz transformation for inertial frames moving with *constant velocities*

relative to one another. Our starting point for modification of Franklin transformation is its main formal difference from the Lorentz transformation which is the dependence of relative velocity on the radial coordinate (i.e $v = (v(r))$ in (12)). It is clear from Franklin's derivation of (12) that this coordinate-dependent velocity is a direct consequence of applying the relativistic composition law to high rotational velocities. Indeed the nonlinear velocity relation (12) could be obtained by the requirement that for any two infinitesimally close points on a uniformly rotating rod (divided into n infinitesimal segments) with angular frequency Ω the linear velocity is given by $\Omega\Delta r$ and then using the relativistic composition law iteratively to find the velocity at a finite distance along the rod in the limit $n \rightarrow \infty$ [8]. Although velocity depends on the radius, since the comparison is made between observations of two frames, an inertial non-rotating one (laboratory observer) and a non-inertial rotating frame at a given radius R , going back to the transformation law (by formal analogy with LT) the velocity at that radius i.e $v = c \tanh(R\Omega/c)$ should enter the transformation law. Indeed it has already been pointed out in some literature [9], without further clarification, that Franklin transformation leads to inconsistencies if one neglects the fact that it is determined at $r = \text{constant}$ as well as $z = \text{constant}$. We have mentioned some of these inconsistencies in previous sections and so by the above argument we introduce the following modified version of Franklin transformation (MFT)

$$\begin{aligned} t' &= \cosh(\Omega R/c)t - \frac{R}{c} \sinh(\Omega R/c)\phi \quad ; \quad r' = r \\ \phi' &= \cosh(\Omega R/c)\phi - \frac{c}{R} \sinh(\Omega R/c)t \quad ; \quad z' = z \end{aligned} \quad (23)$$

This is indeed a simple modification but with profound consequences. Actually we have changed the anholonomic coordinate transformation (13) into the above holonomic transformation. To see its effect, first of all we find the equivalent metric by finding the inverse differential transformation which is

$$\begin{aligned} dt &= \cosh(\Omega R/c)dt' + \frac{R}{c} \sinh(\Omega R/c)d\phi' \quad ; \quad dr = dr' \\ d\phi &= \cosh(\Omega R/c)d\phi' + \frac{c}{R} \sinh(\Omega R/c)dt' \quad ; \quad dz = dz' \end{aligned} \quad (24)$$

and substituting them in the inertial observer's flat spacetime metric (4) we end up with (taking $\beta = \frac{\Omega R}{c}$)

$$ds^2 = c^2 \cosh^2 \beta \left(1 - \frac{r^2}{R^2} \tanh^2 \beta\right) dt'^2 - dr^2 - r^2 \cosh^2 \beta$$

$$(1 - \frac{R^2}{r^2} \tanh^2 \beta) d\phi'^2 + 2cR \sinh \beta \cosh \beta (1 - \frac{r^2}{R^2}) dt' d\phi' - dz^2 \quad (25)$$

Note that now there is a radial coordinate r as well as a constant radius R which specifies a class of observers sitting at that radius. This will allow a kinematical interpretation of the above modified Franklin transformation. In other words no matter what the constant radius in (23), this transformations gives a prescription of how the temporal (t & t') and angular (ϕ & ϕ') coordinates of an event in the two frames are related. Indeed, it is now that one could justify the division of the originally introduced transformation of *arc lengths* ($r'\phi'$ & $r\phi$) by the common radial coordinate leading to the transformation of *angular coordinates* ϕ and ϕ' . It should be noted that spatial coordinate measurements by the inertial as well as the rotating observers are made from the axis of rotation. The presence of R as a constant in the transformed flat spacetime as given by (24) may look strange but obviously it is no stranger than the appearance of Ω in (1) or in (13). Both Ω and R represent transformation to a new frame, one (Ω) from inertial frame to a centrally rotating (spinning) frame and the other (R) from the centrally rotating (spinning) frame to a set of equivalent rotating frames at radius R (orbiting frames). Further it should not be forgotten that the *spacetime* in rotating frame always stays flat whether it is obtained through Franklin transformation or its modified version given by (24) and it is only the *spatial metric* which looses its Euclidean character both in FT and MFT. Obviously the metric (25) is of interest for radial distances

$$r \leq \frac{\beta}{|\tanh \beta|} (\frac{c}{\Omega}), \quad (26)$$

and in the classical Galilean limit where $\beta \ll 1$ (i.e close to the rotation axis) it reduces to

$$ds^2 = c^2(1 - \frac{r^2\Omega^2}{c^2})dt'^2 - dr^2 - r^2(1 - \frac{R^2}{r^2}\beta^2)d\phi'^2 + 2R^2\Omega(1 - \frac{r^2}{R^2})dt'd\phi' - dz^2, \quad (27)$$

which in turn reduces to the spacetime (5) under the extra condition that the radial coordinates of the events under consideration are larger than or equal to R . In other words for observers close to the axis the range $R \leq r < \frac{c}{\Omega}$ replaces the range $0 \leq r < \frac{c}{\Omega}$ [23]. So unlike the Franklin transformation not only the transformation itself but also the metric in rotating frame reduces to the Galilean one in the limit $\beta \ll 1$. It should be noted that for $r = R$ in (25), i.e at the radial position of the rotating observer, the metric reduces to that of spatially Euclidean flat space (5) of an inertial observer i.e at $r = R$ the form of the

spacetime metric is invariant. In other words for the rotating observer sitting at radius R there will be no length contraction of spatial distances composed of arc lengths of a circle at that radius. One could think of observers at different radii as the centrally rotating one who has moved radially to that radius and measures the same proper length for arc lengths without any contraction. Now setting $R = 0$ in (23) and its reduction exactly to GRT while (25) reduces to (4) has a consistent interpretation (in contrast to setting $r = 0$ in FT which was shown to lead to contradiction with respect to its kinematical transformation); it corresponds to the centrally rotating (inertial) observer (spinning observer) who is at rest with respect to the non-rotating inertial observer there, and so their observations are naturally related through GRT. So in our setting of the problem of rotation and rotating frames, we have drastically changed the scenario by introducing non-inertial observers sitting at non-zero radii and introducing the MFT as the kinematical transformation between these observers and the inertial ones. In the next two subsections we find out how the spatial metric and angular velocity of a test particle/disk point are modified in a rotating frame through MFT.

A. Spatial line element and spatial distances

Using the 1 + 3 approach (Appendix A), the metric (25) could be written in the following form

$$ds^2 = c^2 \cosh^2 \beta \left(1 - \frac{r^2}{R^2} \tanh^2 \beta\right) (dt' - A_\alpha dx'^\alpha)^2 - dl^2 \quad (28)$$

in which the spatial line element is given by

$$dl^2 = dr^2 + dz^2 + \left(r^2 \cosh^2 \beta \left(1 - \frac{R^2}{r^2} \tanh^2 \beta\right) + R^2 \frac{\sinh^2 \beta \left(1 - \frac{r^2}{R^2}\right)^2}{\left(1 - \frac{r^2}{R^2} \tanh^2 \beta\right)} \right) d\phi'^2 \quad (29)$$

and the gravitomagnetic potential is

$$A_\alpha \equiv A_{\phi'} \delta_\alpha^{\phi'} = R \frac{\tanh \beta \left(1 - \frac{r^2}{R^2}\right)}{\left(1 - \frac{r^2}{R^2} \tanh^2 \beta\right)} \quad (30)$$

Now one could find the circumference of a circle/disk of radius r in $z = \text{constant}$ plane using the above line element as

$$L_{MFT} = \int dl = \int_0^{2\pi} \left(r^2 \cosh^2 \beta \left(1 - \frac{R^2}{r^2} \tanh^2 \beta\right) + R^2 \frac{\sinh^2 \beta \left(1 - \frac{r^2}{R^2}\right)^2}{\left(1 - \frac{r^2}{R^2} \tanh^2 \beta\right)} \right)^{1/2} d\phi' \quad (31)$$

It is an easy task to show that the above spatial line element reduces to the classical spatial element (8) in the limit of $\beta \ll 1$. Also it is noted that for an observer at the center of the disk i.e $R = 0$ the circumference is equal to $2\pi r$ and for an observer sitting at non-zero radius R a circle at that radius i.e $r = R$, has the Euclidean circumference $2\pi R$ as expected from the form-invariance of the metric (25) at that radius.

B. Angular velocity of a test particle/disk point in the two frames using MFT

In terms of the kinematical interpretation, the angular velocities of a test particle in the two frames related by the MFT is found by employing the inverse differential rotation (24) so that

$$\omega = \frac{d\phi}{dt} = \frac{\cosh \beta d\phi' + \frac{c}{R} \sinh \beta dt'}{\cosh \beta dt' + \frac{R}{c} \sinh \beta d\phi'} \quad (32)$$

leading to

$$\omega' = \omega \left(1 + \frac{R}{c} \tanh \beta\right) - \frac{c}{R} \tanh \beta \quad (33)$$

in which we used the fact that $\omega' = \frac{d\phi'}{dt'}$. As in the case of FT, it could be easily seen that in the limit of $\beta \ll 1$ the above relation reduces to the classical relation (3) which was found through the Galilean transformation. In terms of a rotating disk interpretation, from an inertial observer's point of view, all points on the disk have the angular velocity $\frac{d\phi}{dt} = \Omega$ and so the above relation changes into

$$\omega' = \Omega \left(1 + \frac{R}{c} \tanh \beta\right) - \frac{c}{R} \tanh \beta \quad (34)$$

in other words in MFT, unlike the case of Galilean transformation, for rotating observers the angular velocities of points on a rotating disk are not zero and depend on the radius. But the Galilean expectation is restored in the limit of $\beta \ll 1$ where $\omega' = 0$.

VI. NON-INVARIANCE OF ELECTROMAGNETISM UNDER FRANKLIN TRANSFORMATION

In some literature [10] discussing the Franklin transformation it is claimed that this transformation restores the full Lorentz (-type) covariance of electrodynamics. Here we

show in detail that such a claim is not correct and is obviously in conflict with the spirit of special relativity encoded in Feynman's quote. Under Lorentz transformation Maxwell equations are invariant in the sense that they retain the same 3-dimensional vector form in the transformed coordinates, consequently the electromagnetic wave equation which is obtained from these equations is also form invariant. In what follows we show that neither the Maxwell equations nor wave equation are form-invariant under Franklin transformation. To make life easier we show this in the absence of any EM sources and for the modified Franklin transformation but the same result (non invariance of electromagnetism) holds for the original Franklin transformation. From modified Franklin transformation (23) we have the following relation between the partial derivatives

$$\begin{aligned}\frac{\partial}{\partial t'} &= \cosh \beta \frac{\partial}{\partial t} + \frac{1}{R} \sinh \beta \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \phi'} &= R \sinh \beta \frac{\partial}{\partial t} + \cosh \beta \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial r'} &= \frac{\partial}{\partial r} \quad ; \quad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}\end{aligned}\tag{35}$$

A. non-invariance of wave equation under MFT

Using the above relations the wave equation in the unprimed coordinates (inertial observer)

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{1}{r} \partial_r \left(r \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} - \frac{\partial^2 \psi}{\partial z^2} = 0\tag{36}$$

transforms into

$$\begin{aligned}\left(\frac{r^2 \cosh^2 \beta - R^2 \sinh^2 \beta}{r^2} \right) \frac{\partial^2 \psi}{\partial t'^2} + 2 \left(\frac{(R^2 - r^2) \sinh \beta \cosh \beta}{R r^2} \right) \frac{\partial^2 \psi}{\partial t' \partial \phi'} - \frac{1}{r} \partial_r \left(r \frac{\partial \psi}{\partial r} \right) \\ + \left(\frac{r^2 \sinh^2 \beta - R^2 \cosh^2 \beta}{R^2 r^2} \right) \frac{\partial^2 \psi}{\partial \phi'^2} - \frac{\partial^2 \psi}{\partial z^2} = 0\end{aligned}\tag{37}$$

under MFT, i.e the wave equation is not form-invariant under MFT. The same result could also be obtained by using the metric corresponding to MFT (Eqn. (25)) and the following general form of the wave equation in a curved background with metric g_{ij}

$$\square \psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \left(g^{1/2} g^{ik} \frac{\partial \psi}{\partial q_k} \right) = 0\tag{38}$$

where $q_i = t', r, \phi', z$.

B. non-invariance of Maxwell equations under MFT

To obtain Maxwell equations for a rotating observer from those in the frame of an inertial observer related through MFT we use the field tensor in the spacetime of a rotating observer (MFT metric) given by [12] :

$$F'_{ij} = \begin{pmatrix} 0 & -\frac{A}{R}E'_r & -rE'_{\phi'} & -\frac{A}{R}E'_z \\ \frac{A}{R}E'_r & 0 & -\frac{\tilde{A}}{A}E'_r + \frac{Rr}{A}B'_z & -B'_{\phi'} \\ rE'_{\phi'} & \frac{\tilde{A}}{A}E'_r - \frac{Rr}{A}B'_z & 0 & \frac{\tilde{A}}{A}E'_z + \frac{Rr}{A}B'_r \\ \frac{A}{R}E'_z & B'_{\phi'} & -\frac{\tilde{A}}{A}E'_z - \frac{Rr}{A}B'_r & 0 \end{pmatrix} \quad (39)$$

where

$$A = \sqrt{R^2 \cosh^2 \beta - r^2 \sinh^2 \beta} \quad \text{and} \quad \tilde{A} = (-R^2 + r^2) \sinh \beta \cosh \beta \quad (40)$$

so that the inhomogeneous equations (in the absence of sources)

$$\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} F'^{ij}) = 0 \quad (41)$$

are given by

$$\partial_r [r(\frac{R}{A}E'_r - \frac{\tilde{A}}{rA}B'_z)] + \partial_{\phi'}(E'_{\phi'}) + \partial_z [r(\frac{R}{A}E'_z + \frac{\tilde{A}}{A}B'_r)] = 0 \quad (42)$$

$$\frac{R}{A}\partial_{t'}E'_r - \frac{\tilde{A}}{rA}\partial_{t'}B'_z - \frac{A}{rR}\partial_{\phi'}B'_z + \partial_z B'_{\phi'} = 0 \quad (43)$$

$$\partial_{t'}E'_{\phi'} + \partial_r(\frac{A}{R}B'_z) - \partial_z(\frac{A}{R}B'_r) = 0 \quad (44)$$

$$\frac{rR}{A}\partial_{t'}E'_z + \frac{\tilde{A}}{A}\partial_{t'}B'_r - \partial_r(rB'_{\phi'}) + \frac{A}{R}\partial_{\phi'}B'_r = 0 \quad (45)$$

respectively for $j = 0, 1, 2, 3$. Also the homogeneous equations

$$\partial_{[i}F'_{jk]} = 0 \quad (46)$$

give rise to

$$\partial_r(\frac{\tilde{A}}{A}E'_z + \frac{Rr}{A}B'_r) + \partial_{\phi'}B'_{\phi'} + \partial_z(-\frac{\tilde{A}}{A}E'_r + \frac{Rr}{A}B'_z) = 0 \quad (47)$$

$$\partial_{t'}(-\frac{\tilde{A}}{A}E'_r + \frac{Rr}{A}B'_z) + \partial_r(rE'_{\phi'}) - \partial_{\phi'}(\frac{A}{R}E'_r) = 0 \quad (48)$$

$$\partial_{t'}B'_{\phi'} - \partial_r(\frac{A}{R}E'_z) + \partial_z(\frac{A}{R}E'_r) = 0 \quad (49)$$

$$\frac{\tilde{A}}{A}\partial_{t'}E'_z + \frac{rR}{A}\partial_{t'}B'_r + \frac{A}{R}\partial_{\phi'}E'_z - \partial_z(rE'_{\phi'}) = 0 \quad (50)$$

These equations are different in form from those obtained in the non-rotating inertial frame which are given by the above equations with $A = R$ and $\tilde{A} = 0$.

The same results as above could also be obtained by first using the field tensor in flat spacetime in cylindrical coordinates;

$$F_{ij} = \begin{pmatrix} 0 & -E_r & -rE_\phi & -E_z \\ E_r & 0 & rB_z & -B_\phi \\ rE_\phi & -rB_z & 0 & rB_r \\ E_z & B_\phi & -rB_r & 0 \end{pmatrix} \quad (51)$$

to write the maxwell equations in the non-rotating inertial frame and then employ the general relation between the field tensors in the two frames

$$F_{ij} = \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^n}{\partial x^j} F'_{mn} \quad (52)$$

to relate the primed and unprimed electromagnetic fields and finally replace the unprimed quantities (including partial differentials using Eqn. (35)) by the primed ones. So in general neither wave equation nor the Maxwell equations are invariant under MFT.

VII. DISCUSSION AND SUMMARY

We have discussed a proposed relativistic rotational transformations (dubbed as Franklin transformation) relating coordinates of an inertial non-rotating frame to the one rotating around their common axis with constant angular velocity Ω (measured by the inertail observers). Advantages and also drawbacks of this transformation specially with respect to the spacetime metric from the rotating observer's point of view as well as with its kinematical interpretation are pointed out. By introducing non-inertail observers sitting at non-zero radii we have modified FT from a non-holonomic transformation to a holonomic one and showed how the modified transformation gives rise to a more consistent spacetime metric for these observers. The resulted spacetime metric includes two parameters Ω and R , corresponding to the rotational angular velocity and radial position of these observers and although flat, has a non-Euclidean spatial line element (found through 1 + 3 formulation of spacetime decomposition) leading to non-Euclidean value for the circumference of a rotating disk or any other circle of a given radius. In our setting of the problem of relativistic rotational transformation there are three different kinds of observers for a given angular

velocity: **I**-Inertial non-rotating observers **II**-centrally rotating observer (spinning observer) and **III**-Non-inertial rotating observers at non-zero radii (orbiting observers). At the position of orbiting observers (i.e at $r = R$) the spacetime metric is found to be invariant and spatially Euclidean, a fact hinting toward a possible relation with Fermi metric. Indeed this is an interesting evidence reinforcing our interpretation of the MFT and the corresponding metric which could be provided by studying the relation between the metric (25) and the Fermi metric. These matters will be discussed elsewhere [13]. It is also shown explicitly that, against the claims made, neither Maxwell equations nor wave equation are invariant under FT or MFT.

From observational point of view it is expected that application of a relativistic rotational transformation to modify physical effects related to rotating systems and phenomena. Some of the example include transverse Doppler effect, Sagnac effect and rotational properties of pulsars. Transverse Doppler effect, for a source circling a receiver on a rotating disk, will be affected naturally due to the nonlinear velocity (12) introduced in FT. Also it is expected that employing a relativistic rotational transformation will lead to a relativistic Sagnac effect where this relativistic effect is distinct from the one due to propagation of light in a medium where relativistic velocity addition rule applies. Finally, fastest rotating celestial objects are pulsars and the fastest pulsar, named PSR J1748-2446ad is located some 28,000 light-years from Earth in the constellation Sagittarius and is spinning 716 times per second, or at 716 Hertz (Hz). If its radius is taken to be 16 km it will have a linear velocity of 75000k km/s i.e about %25 that of light speed at the equator. It is expected that at this rotational velocity a relativistic rotational transformation to be at work and observationally effective. The above physical effects/phenomena clearly indicate that a relativistic rotational transformation such as FT or its modified version (MFT) discussed here deserves more attention.

Appendix A: 1 + 3 (threading) formulation of spacetime decomposition

Unlike the 3 + 1 formulation of spacetime decomposition in which spacetime is foliated into constant-time hypersurfaces, in the 1 + 3 formulation it is decomposed into threads tracking history of each spatial point. This formulation of spacetime decomposition starts from the following general form for the spacetime metric [14]

$$ds^2 = g_{00}(dx^0 - A_\alpha dx^\alpha)^2 - dl^2, \quad (\text{A1})$$

in which $A_\alpha = -\frac{g_{0\alpha}}{g_{00}}$ is the so-called gravitomagnetic potential and

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta = (-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}) dx^\alpha dx^\beta \quad \alpha, \beta = 1, 2, 3 \quad (\text{A2})$$

is the spatial line element (also called the radar distance element) in terms of the three-dimensional spatial metric $\gamma_{\alpha\beta}$. It is the integral of this line element which gives the spatial distance between two events,

$$L = \int_{x_i^\alpha}^{x_f^\alpha} dl \quad (\text{A3})$$

For two simultaneous events at nearby points x^α and $x^\alpha + dx^\alpha$ the difference between their coordinate (world) time is given by

$$\Delta x^0 = A_\alpha dx^\alpha, \quad (\text{A4})$$

This allows one to synchronize clocks in an infinitesimal region of space and also along any open curve. But synchronization of clocks along a closed path is not possible in general, since upon returning to the initial point the world time difference is not zero and in the case of stationary spacetimes is given by the line integral

$$\Delta x^0 = \oint A_\alpha dx^\alpha \quad (\text{A5})$$

taken along the closed path. Using the above equation the world-time difference for two photons started at the same point but travelling in opposite directions (clockwise and counter clockwise) along a circle of radius R on a disk rotating with angular velocity Ω such that $\frac{\Omega R}{c} \ll 1$ is given by

$$\Delta t = 4\pi R^2 \frac{\Omega}{c^2} \quad (\text{A6})$$

This difference which leads to a phase shift $\delta\phi = \frac{2\pi c\Delta t}{\lambda}$ could also be obtained through classical reasoning by an inertial non-rotating observer and is the theoretical basis of the so called Sagnac effect [15] or in its modern version, ring laser interferometry. It is this same formulation of spacetime decomposition which allows one to write Einstein field equations in a quasi-Maxwell form in terms of the *gravitoelectric* and *gravitomagnetic* fields in the context of the so called *Gravitoelectromagnetism*[16].

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- [1] R. P. Feynman, Feynman lectures on physics, Vol. 2, 1968.
 - [2] Relativity in Rotating Frames: Relativistic Physics in Rotating Reference Frames Eds. G. Rizzi and M. L. Ruggiero, Kluwer academic publishers, 2004.
 - [3] P. Ehrenfest, Physik, Zeits. 10, 918 (1909).
 - [4] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, New York, 1995.
 - [5] Ph. Franklin, Proc. Nat. Acad. Sci., volume 8, No. 9, 1922.
 - [6] M. G. Trocheris, Philos. Mag. **40**, 1143 (1949).
 - [7] H. Takeno, Prog. Theor. Phys. **7**, 367 (1952).
 - [8] Z. X. Cao, Ch.L. Chen and L. Liu, arXiv:physics/0304006v2 [physics.gen-ph].
 - [9] S. Kichenassamy and R. K. Krikorian, the astrophysical journal, 431 : 715-717, 1994.
 - [10] S. Kichenassamy and R. K. Krikorian, J. Math. Phys. **35**, 5726 (1994).
 - [11] P. Hillion, Phys. Rev. E, **57**, 6, 7239 (1998).
 - [12] Nouri-Zonoz, M. and Ramezani-Aval, H., Electromagnetic field tensor and field equations in curved spaces and spacetimes, in preparation .
 - [13] Nouri-Zonoz, M. and Ramezani-Aval, H., On rotation nd rotating frames II: Modified Franklin transformation and Fermi metric, in preparation.
 - [14] Landau, L. D. and Lifshitz, E. M., *The classical theory of fields*, Pergamon Press, 1975.

- [15] G. Sagnac, Comptes Rendus 157: 708710 (1913)
- [16] Lynden-Bell, D. and Nouri-Zonoz, M., Rev. Mod. Phys. 70, 427 (1998).
- [17] For a recent discussion on this subject refer to [2].
- [18] It should be remembered that these observers are able to discover the rotation either by mechanical (Focult pendulum) or electromagnetic (Sagnac effect) means and therefore assign a linear velocity to the rim and consequently decide that $P' < P$ where P is the circumference of a non-rotating disk measured before the disk was set to rotate uniformly.
- [19] In other words Rindler space in the limit $x^1 \ll 1$ (i.e for points infintesimally close to the world line of the observer) reduces to the Fermi metric [4] in the absence of rotation (i.e $\Omega = 0$) while (5) in the limit $r \ll 1$ (i.e infintesimally close to the centrally rotating observer) reduces to the Fermi metric in the absence of linear acceleation (i.e $a = 0$).
- [20] By *points* one could also mean *observers* at that point and as we will discuss later this depends on the interpretation of the transformation.
- [21] In some literature this transformation is called Takeno-Trocheris transformation, but due to Franklin's precedence by almost 30 years and also to highlight his largely ignored work, here we will call it *Franklin transformation*.
- [22] It should be noted that Ω is taken as a constant and such that the integrity of the rotating disk is retained.
- [23] Note that the condition $\beta \ll 1$ is equivalent to $R \ll \frac{c}{\Omega}$ whereas the same condition employed in (26) leads to $r \leq \frac{c}{\Omega}$.